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ON FILLINGS OF HOMOTOPY EQUIVALENT CONTACT STRUCTURES

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Abstract. This paper provides a topological method for filling contact structures on the connected sums of $S^2 \times S^3$. Examples of nonsymplectomorphic strong fillings of homotopy equivalent contact structures with vanishing first Chern class on $\#_k S^2 \times S^3$ ($k \geq 2$) are produced.

0. Introduction

The study of four dimensional topology has seen great advances for the last 35 years after Freedman ([7]) and Donaldson ([1]). In the last several years, exotic smooth structures on rather small 4-manifolds have been discovered ([6]). This note exploits the symplectic structures on these simply connected 4-manifolds with small second homology groups. The fillings of contact 5-manifolds are distinguished by the lifts of symplectic surfaces inside the 4-manifolds into the fillings. In general, fillings of contact manifolds may be used to extract information about the contact structure. In this paper the contact structures on the filled 5-manifold are homotopy equivalent. However it is not clear whether these contact structures are isotopic or contactomorphic.

The paper consists of two sections. Section 1 is on preliminaries about contact structures and fillings of them. The second section includes the main theorems (Theorem 2.2 and Theorem 2.6) and their proofs.

1. Preliminaries

This section reviews some definitons and facts about contact structures on 5-manifolds. More information can be found in ([8]).

Definition 1.1. *Let N be a manifold of odd dimension $2n+1$. A contact structure is a maximally nonintegrable hyperplane field $\xi = kernel(\alpha) \subset TM$. The defining differential 1-form α is required to satisfy $\alpha \wedge (d\alpha)^n > 0$. Such a 1-form α is called a contact form. The pair (N, ξ) is called a contact manifold. A complex bundle structure J on ξ is called ξ-compatible if $J_p : \xi_p \to \xi_p$ is a $d\alpha$-compatible complex structure on ξ_p for each point $p \in N$, where α is any contact form such that $\xi = kernel(\alpha)$. A ξ-compatible almost contact structure on a contact manifold (N, ξ) is a complex structure on ξ which is ξ-compatible.*

The condition $\alpha \wedge (d\alpha)^n > 0$ means that the orientation of N and the orientation imposed by the contact structure are same. N is oriented by $\alpha \wedge (d\alpha)^n > 0$ and ξ is oriented by $(d\alpha)^{n-1}$. Note that $d\alpha$ gives a symplectic vector bundle structure to ξ.

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Definition 1.2. *Two contact manifolds (N_0, ξ_0) and (N_1, ξ_1) are said to be contactomorphic if there is an orientation preserving diffeomorphism $f : N_0 \to N_1$ with $df(\xi_0) = \xi_1$, where $df : TN_0 \to TN_1$ denotes the differential of f. If $\xi_i = kernel(\alpha_i)$, $i = 0, 1$, this is equivalent to saying that α_0 and $f^*\alpha_1$ determine the same hyperplane field, and hence equivalent to the existence of a positive function $g : N_0 \to \mathbb{R}^+$ such that $f^*\alpha_1 = g\alpha_0$.*

Two contact structures ξ_0 and ξ_1 on a smooth manifold N are said to be homotopy equivalent if their respective almost contact structures are homotopy equivalent. ξ_0 and ξ_1 are said to be isotopic if there is a smooth isotopy ψ_t $(t \in [0, 1])$ of N such that $T\psi_t(\xi_0) = \xi_t$ for each $t \in [0, 1]$. Equivalently, $\psi_t^*\alpha_t = \lambda_t\alpha_0$, where $\lambda_t : N \to \mathbb{R}^+$ is a suitable smooth family of smooth functions. This is equivalent to existence of a contactomorphism $f : (N, \xi_0) \to (N, \xi_1)$ which is isotopic to the identity.

If two contact structures ξ_0 and ξ_1 on N are isotopic then (N, ξ_0) and (N, ξ_1) are contactomorphic. Homotopy equivalence is much weaker than the isotopy and there may be many nonisotopic contact structures in a homotopy type.

A simply connected 5-manifold N admits an almost contact structure if and only if its integral Stiefel-Whitney class W_3 vanishes. Homotopy classes of almost contact structures are in one to one correspondence with integral lifts of $w_2(TN)$. The correspondence is given by associating to an almost contact structure its first Chern class ([8] p368).

Definition 1.3. *A compact symplectic manifold (M, ω) is called a strong (symplectic) filling of (N, ξ) if $\partial M = N$ and there is a Liouville vector field Y defined near ∂M, pointing outwards along ∂M, and satisfying $\xi = kernel(\omega(Y, \cdot)|_{TM})$ (as cooriented contact structure). In this case we say that (N, ξ) is the convex (or more precisely: ω-convex) boundary of (M, ω).*

For contact manifolds of dimensions greater than three, M is a strong filling of N if and only if $\partial M = N$ as oriented manifolds and $\omega|_\xi$ is in the conformal class of $d\alpha|_\xi$ (Theorem 5.1.5 of [8]). The boundary of a strong filling is said to be of contact type.

Let (X, ω) be a symplectic 4-manifold and e be a second cohomology class of X. Let's denote the 2-disk bundle over X with Euler class e by M_e. Let π be the projection map of the fibration. Any symplectic form on M_e is locally $\pi^*\omega \oplus \omega_a$ where ω_a is the symplectic structure on the fiber for $a \in X$. Any two such symplectic forms agree on the zero section, therefore M_e has a symplectic structure which is unique up to symplectomorphism by the symplectic neighborhood theorem ([10]). The next lemma is deduced from [3].

Lemma 1.4. *Let $[\omega]$ be the cohomology class of the symplectic form. If $[\omega]e < 0$ then the contact structure on the boundary is of contact type.*

2. Nonsymplectomorphic Fillings of a Homotopy Type

Assume that X is a closed, simply connected, smooth 4-manifold and $e \in H^2(X; \mathbb{Z})$ is a primitive, characteristic class. If X_e is the total space of the S^1-bundle over X with Euler class e, then X_e is diffeomorphic to $\#_{b_2(X)-1} S^2 \times S^3$ ([2]).

The pullback of the almost contact structure which is compatible with $d\alpha$ on the 5-manifold is the pullback of the symplectic form ω on the base 4-manifold. The next lemma is relating the first Chern class of the contact structure on the boundary and the symplectic structure on the base 4-manifold.

Lemma 2.1. *The first Chern class of a compatible almost contact structure is the pullback of the first Chern class of the symplectic structure ω on X.*

2.1. Fillings of $\#_k S^2 \times S^3$ $(k \geq 2)$.

Theorem 2.2. *In the homotopy equivalence class of contact structures on $\#_2 S^2 \times S^3$ with the first Chern class equal to zero, there are contact structures which have nonsymplectomorphic strong fillings.*

Proof. According to Fintushel and Stern ([5,6]), there are infinitely many mutually nondiffeomorphic smooth manifolds which are homeomorphic to $\mathbb{C}P^2 \#_2 \overline{\mathbb{C}P^2}$, two of which carry symplectic structures. Let (X_0, ω_0) be $\mathbb{C}P^2 \#_2 \overline{\mathbb{C}P^2}$ and let (X_1, ω_1) be the symplectic 4-manifold which is homeomorphic to $\mathbb{C}P^2 \#_2 \overline{\mathbb{C}P^2}$, but not diffeomorphic to it as given in [6]. X_0 is not minimal and X_1 is minimal. Both of the manifolds have just two Seiberg-Witten basic classes which are plus and minus the canonical class. The first Chern class of X_0 is $3H - E_1 - E_2$. On the other hand the canonical class of X_1 evaluates positive with the symplectic form, because it is a surface of general type ([6] page 66). The first Chern class of X_1 is either $3H - E_1 - E_2$ or $-(3H - E_1 - E_2)$.

Let e_0 be $-c_1(X_0) = -(3H - E_1 - E_2) \in H^2(X_0; \mathbb{Z})$ and e_1 be $c_1(X_1)$ in $H^2(X_1; \mathbb{Z})$. For $j = 0, 1$, the boundary of M_{e_j} is the circle bundle over (X_j, ω_j) with Euler class e_j, and the smooth structures on the boundaries of M_{e_1} and M_{e_2} are diffeomorphic to $\#_2 S^2 \times S^3$ ([2]). Since X_j is simply connected, a part of the Gysin sequence for this circle bundle over X_j is as shown below.

$$(1) \quad 0 \to H^0(X_j; \mathbb{Z}) \stackrel{\cup e_j}{\to} H^2(X_j; \mathbb{Z}) \stackrel{\pi^*}{\to} H^2(\#_{b_2(X_j)-1} S^2 \times S^3; \mathbb{Z}) \to 0 = H^1(X_j; \mathbb{Z})$$

The image of the the map $\cup e_j$ is generated by e_j that is by plus or minus $c_1(X_j)$. By Lemma 2.1, the pullbacks of $c_1(\xi_0)$ and $c_1(\xi_1)$ on $\#_2 S^2 \times S^3$ are the first Chern classes of the respective symplectic structures on X_0 and X_1. These classes are in the kernel of π^*, therefore the first Chern classes of the respective contact structures are zero.

The symplectic form ω_0 on X_0 couples negatively with e_0 ([9]) and $\omega_1 \cdot e_1$ is less than zero by the discussion above. By Lemma 1.4 the boundaries are of contact type. The boundaries are diffeomorphic to $\#_2 S^2 \times S^3$ and the first Chern classes of the corresponding contact structures are zero.

It remains to show that the symplectic structures are different. This is done by a count of J-holomorphic curves. For $j = 0, 1$ the inclusion of X_j into M_{e_j} induces an injection of $H_2(X_j; \mathbb{Z})$ into $H_2(M_{e_j}; \mathbb{Z})$. Let $\overline{E}$ be the image of $E \in H_2(X_j; \mathbb{Z})$ in $H_2(M_{e_j}; \mathbb{Z})$. Remember $E \cdot E$ is -1 in X_j. Let J_j be a generic almost complex structure which is compatible with the symplectic structure on M_{e_j}. Since E is the class of an exceptional sphere in X_0, E has an almost complex sphere representative in X_0. The image of this sphere under the inclusion map is a J_0-holomorphic sphere representative of $\overline{E}$ in M_{e_j}. Assume $\overline{E}$ has an J_1-holomorphic sphere M_{e_1} that represents $\overline{E}$. Then E would have a sphere representative in X_1. But this is not the case because X_1 is minimal. Therefore M_{e_0} and M_{e_1} are not symplectomorphic. $\square$

Remark 2.3. *It is not clear for the author that whether the symplectic structures on the 4-manifolds X_0 and X_1 are related in any way. As a result of the reverse engineering which is applied to a model manifold, X_1 is known to be symplectic (*[4], [5]*). In* [5] *and* [6] *this symplectic manifold X_1 is obtained from X_0 with a surgery on a single nullhomologous torus. But the latter operation does not involve the symplectic structures. So there is an ambiguity in the choice of e_1 in the proof.*

Remark 2.4. *In* [13]*, Stipsicz and Szabo note that SeibergWitten invariants can tell apart only at most finitely many symplectic structures on the topological manifold $\mathbb{C}P^2\#_k\overline{\mathbb{C}P^2}$ with $k \leq 8$. Therefore this infinity result can not be extended to lower k in an obvious way by the methods of this paper.*

By using contact surgery and symplectic handlebody results of Meckert and Weinstein ([11, 14]) one can say:

Corollary 2.5. *For $k \geq 2$, in the homotopy equivalence class of contact structures on $\#_kS^2 \times S^3$ with the first Chern class equal to zero, there are contact structures which have nonsymplectomorphic strong fillings.*

2.2. **Fillings of $\#_kS^2 \times S^3$ ($k \geq 9$).** Dolgachev surfaces are elliptic surfaces that are homeomorphic to the elliptic surface $E(1)$ but not diffeomorphic to it. These manifolds are denoted by $E(1)_{p,q}$. $E(1)_{p,q}$ can be constructed from $E(1)$, which is diffeomorphic to $\mathbb{C}P^2\#_9\overline{\mathbb{C}P^2}$, by p and q logarithmic transformations where $gcd(p, q) = 1$ and $p > q > 1$. Considering the infinitely many different symplectic structures on these manifolds, one can conclude as follows.

Theorem 2.6. *In the homotopy equivalence class of contact structures on $\#_9S^2 \times S^3$ with the first Chern class equal to zero, there are contact structures which have infinitely many nonsymplectomorphic strong fillings.*

Proof. Assume that $p, q \in \mathbb{Z}$ such that $gcd(p, q) = 1$ and $p > q > 1$. If F is the class of a generic fiber of the elliptic fibration on $E(1)_{p,q}$, then there is a homology class $A_{p,q} = \frac{F}{pq}$ in $H_2(E(1)_{p,q}; \mathbb{Z})$. The first Chern class of $E(1)_{p,q}$ is $-(pq - p - q)PD(A_{p,q})$, a negative multiple of Poincare dual of $A_{p,q}$. Let $e_{p,q}$ be $-PD(A_{p,q})$, which is primitive, and $M_{p,q}$ be the total space of the disk bundle over $E(1)_{p,q}$ with Euler class $e_{p,q}$. The boundary of $M_{p,q}$ is the circle bundle $X_{p,q}$ over $E(1)_{p,q}$ with Euler class $e_{p,q}$ and it is diffeomorphic to $\#_9S^2 \times S^3$ for all p, q.

By Lemma 2.1, the pullbacks of the first Chern classes of the contact structures on $X_{p,q}$ the first Chern classes of the respective symplectic structures on $M_{p,q}$. The Gysin sequence for this circle bundle over $E(1)_{p,q}$ gives the first Chern classes of these contact structures are zero.

$E(1)_{p,q}$ are simply connected (proper) elliptic surfaces. According to the Kodaira-Enriques classification of complex surfaces, the symplectic form evaluates negatively on $c_1(E(1)_{p,q})$ and on $e_{p,q}$. By Lemma 1.4, for each $\{p, q\}$, the symplectic structure on the disk bundle $M_{p,q}$ is strong filling of its contact boundary.

Let $\overline{A}_{p,q} \in H_2(M_{p,q}; \mathbb{Z})$ be the pushforward of the class $A_{p,q} \in H_2(E(1)_{p,q}; \mathbb{Z})$. As explained by Ruan and Tian in [12] page 505, for the choice of a complex structure $J_{p,q}$, among all multiples of $mA_{p,q}$ (for $0 < m < pq$), only $pA_{p,q}$ and $qA_{p,q}$ have connected J_{pq}-holomorphic torus representatives. Moreover this choice of complex structure is generic. This means, for any two distinct couples $\{p, q\}$ and $\{p', q'\}$, either $p\overline{A}_{p,q}$ and $q\overline{A}_{p,q}$ or $p'\overline{A}_{p',q'}$ and $q'\overline{A}_{p',q'}$ have connected complex

torus representatives in the total space of the disk bundle. Therefore symplectic structures on $M_{p,q}$ and $M_{p',q'}$ are not symplectomorphic. $\square$

Corollary 2.7. *For $k \geq 9$, in the homotopy equivalence class of contact structures on $\#_k S^2 \times S^3$ with the first Chern class equal to zero, there are contact structures which have infinitely many nonsymplectomorphic strong fillings.*

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